Natural Sorting over Permutation Spaces

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0. Introduction. In this paper we continue the study, begun in [1], of some combinatorial problems related to monotonicities that occur in certain spaces of finite sequences. These spaces are equipped with standard probability measures, so that one may study the distribution of monotonicities in such spaces and, in particular, the expected lengths of maximal monotonic subsequences. These, in turn, are upper bounds on the expected lengths of monotonic subsequences obtained by applying some selection process over the space of sequences. The problem which we have called *natural sorting* is concerned with the maximization of these expected lengths.

The scheme of the paper is as follows. In Section 1 we give definitions and review some background material. In Section 2 we describe the distribution of maximal sequences occurring in the space of permutation sequences.* These distributions have been computed exactly for spaces of permutations of length $n = 2(1)36$, and have been approximated by Monte Carlo computations for certain values of n ranging up to 10,000. In Section 3 we consider several selection strategies and the corresponding distribution of selected monotonic subsequences.

The computations were performed on the IBM 7094 at the Computer Center of the University of California, Berkeley. We are indebted to David M. Matula for some of the calculations in Section 3. We should like to thank Geri Stephen for her assistance in the preparation of the manuscript.

1. Definitions and Conventions.

1.1. Throughout this paper the term *sequence* is to be understood to mean *finite sequence*. Standard terms, if not here defined, are used according to the definitions given in [3].

1.2. *Definition*. Let $(X, <)$ be a (totally) ordered space and let n be a fixed natural number. A partial ordering is induced in each element of the cartesian product X^n in the following natural way. If $(x_1, \cdots, x_n)$ is in X^n let S denote the space consisting of the set $\{x_1, \cdots, x_n\}$ together with the partial ordering $x_i \leqq x_j$ if and only if $x_i < x_j$ and $i < j$. In what follows, the space S varies over the set of permutation sequences obtained from $\langle 1, 2, \cdots, n\rangle$.

1.3. *Definition*. A *chain* in S is a (totally) ordered subset of S. The length of a chain is the number of elements in it. A *maximal* chain in S is a chain that is not a proper subset of any other chain in S. A maximal chain in S which has length at least as great as that of any other chain in S is called a *spine* of S.

1.4. *Definition*. A maximal chain in S which has length at least as small as that of any other chain in S is called an *Erdös chain*. (For results on Erdös chains, see [1], [8], and [12].)

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* We shall treat cases of other sorting spaces—in particular sorting spaces of binary sequences—separately.

1.5. *Definition.* By an *nth order selection algorithm* (applied to sequences of length n) is meant an algorithm A which selects a monotonic subsequence from a sequence $(x_1, \cdots, x_n)$ according to the following scheme. The first entry x_1 is selected or rejected on the basis of its size according to a rule of A, so we might write, according to $A(x_1)$. The second entry is selected or rejected in a manner determined by the ordered pair (x_1, x_2) so we might write, according to $A(x_1, x_2)$; and here we are to understand that A acts upon knowledge of whether x_1 was selected or not. Similarly x_3 is selected or rejected by A according to $A(x_1, x_2, x_3)$, and here we understand that A has the information as to which (if any) subset of $\{x_1, x_2\}$ was selected. And so, for each x_i, the selection or rejection of x_i is determined by A on the basis of the set $\{x_j : j \leqq i\}$ and upon the selected subset of this set. So an nth order selection algorithm is essentially one in which all information (concerning selections already made and the preceding part of the original sequence) up to the current point of selection is available to the algorithm. The opposite notion is embodied in the definition of *0th order selection algorithm,* which, applied to a sequence $(x_1, \cdots, x_n)$ selects or rejects each x_i $(i \leqq n)$ according to $A(x_i, s)$ where s is the value of the last selection (if there were such) preceding the decision to select x_i; in other words, a 0th order algorithm simply selects or rejects each x_i according to a rule which depends only upon this x_i and the last previously selected entry and is oblivious to the past history of the sequence.

1.6. *Definition.* The *distinguished* (nondecreasing) subsequence of the sequence $(x_1, \cdots, x_n)$ is obtained using the 0th order selection algorithm: Select x_1. For each $i > 1$, select x_i if x_i is not less than the element selected last, prior to consideration of x_i.

2. The Distribution of Monotonicities in P_n. In [1] the authors considered the question of monotonicities in the space P_n consisting of the space of all permutations of $\langle 1, 2, \cdots, n\rangle$. Our viewpoint was to consider simultaneously the maximal increasing and decreasing subsequences of the elements of this space. Some preliminary results concerning this distribution were obtained in [1], and an encompassing result was derived by Schensted [12], who showed the relationship between the distribution and representations of the symmetric group. However, the question of the actual distributions was left open both by [1] and [13]. We now proceed to fill in this gap by exhibiting the results of the requisite computations. First, however, we review the basis of the calculations.

2.1. *Definition.* A Young tableau of order n is an array of the integers $1, 2, \cdots, n$ satisfying the following. The array consists of rows and columns. For each row, the entries in that row form an increasing sequence. For each column, the entries (moving down that column) form an increasing sequence. Each row contains at least as many entries as the row beneath it. Each column contains as least as many entries as the column to its right.

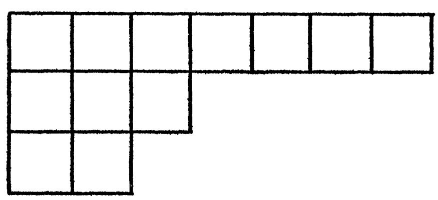
Each permutation of $\langle 1, 2, \cdots, n\rangle$ uniquely determines a Young tableau. The determination proceeds as follows. Let the permutation be $\langle x_1, x_2, \cdots, x_n\rangle$. For the moment, define the first entry in the first row of the tableau to be x_1. Now, if at the ith step, the first i entries of the sequence have been used in the developing tableau then at the next step the element x_{i+1} is inserted into the first row of the tableau by displacing the smallest entry in the first row which is larger than x_{i+1}

or by appending x_{i+1} at the end of the first row if it is larger than all entries in the first row.

If an entry y is displaced from the first row by x_{i+1} then y is inserted into the second row by letting it displace the smallest entry in the second row which is larger than y or by simply appending y to the second row if there is no such element. The process is continued from row to row until either the original x_{i+1} or a displaced element is appended to the end of a row. Then the whole process is renewed for $x_{i+2}, \cdots$ until all of the entries of the original permutation sequence have been entered into the tableau.

2.2. THEOREM (SCHENSTED). *If T is the Young tableau generated by the permutation sequence $\langle x_1, \cdots, x_n \rangle$ then the greatest length of a maximal monotone increasing subsequence of $\langle x_1, \cdots, x_n \rangle$ is equal to the number of columns of T, and the greatest length of a maximal monotone decreasing subsequence of $\langle x_1, \cdots, x_n \rangle$ is equal to the number of rows of T.*

2.3. *Definition.* By a *partition* of the positive integer n we mean a monotonically nonincreasing sequence of positive integers which sum to n. By a *partition tableau* corresponding to a partition $(m_1, \cdots, m_k)$ we mean an upper-left rectangular array of cells (for example, the figure below)



in which there are m_1 cells in the first row, m_2 cells in the second row, $\cdots$, and m_k cells in the kth row. To each cell of a partition array is assigned a number h called its *hook number*. If b is the number of cells below a designated cell and if r is the number of cells to the right of this designated cell, then the value of h corresponding to the designated cell is $h = b + r + 1$.

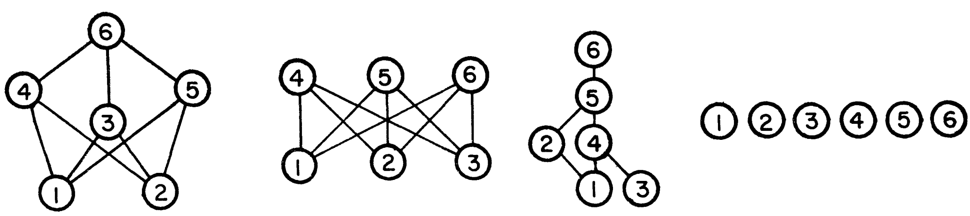
We illustrate these matters in 2.4.

2.4. *Example.* Consider the four sequences:

(i) 2, 1, 5, 4, 3, 6 (ii) 3, 2, 1, 6, 5, 4 (iii) 3, 1, 4, 2, 5, 6 (iv) 6, 5, 4, 3, 2, 1

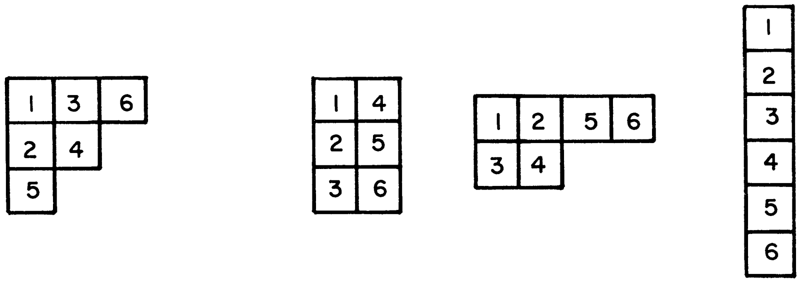
According to our convention, these sequences correspond to the four partially ordered sets with Hasse diagrams [3]:

(i) (ii) (iii) (iv)



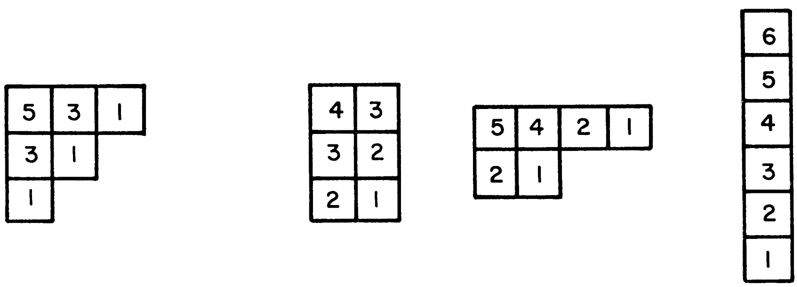
The corresponding Young tableaux are:

(i) (ii) (iii) (iv)



The Young tableaux with their associated hook numbers are:

(i) (ii) (iii) (iv)



$w = [6!/5 \cdot 3^2]^2$ $= 16^2$ $w = [6!/4 \cdot 3^2 \cdot 2^2]^2$ $= 5^2$ $w = [6!/5 \cdot 4 \cdot 2^2]^2$ $= 9^2$ $w = 1$

If one considers the Young tableaux generated by permutations of $(1, 2, \cdots, n)$ it is evident that different permutations can give rise to the same tableau shape (i.e., partition tableau). The number of tableaux with the same shape arising in this fashion is given by a powerful combinatorial theorem:

2.5. THEOREM (FRAME-ROBINSON-THRALL [10]). *The number of tableaux with given shape that contain the integers* $1, 2, \cdots, n$ *is* $n!/\prod_i h_i$, *where the* h_i *are the hook numbers associated with the cells of the tableau.*

Finally we need [12, Theorem 3], which is obtained from 2.2 and 2.5.

2.6. THEOREM (SCHENSTED). *The number of sequences consisting of the distinct numbers* $x_1, \cdots, x_n$ *and having a longest increasing subsequence of length* j *and a longest decreasing subsequence of length* k *is the sum of the squares of the numbers of identically shaped partition tableaux with* j *columns and* k *rows.*

Based upon the preceding results, the distribution of monotonicities has been computed exactly for the spaces p_n $(n \leqq 36)$. The procedure is to generate the distinct partitions of n, to generate the partition tableaux, load the cells with their hook numbers, evaluate the square of the value of the Frame-Robinson-Thrall function. This result is then added to one of several running sums, corresponding to the maximal partition element m_1 (for the increasing subsequences of greatest length where this length happens to be m_1) or to one of several running sums corresponding to k (for the case of the decreasing subsequences of greatest length where this length happens to be k, and $(m_1, \cdots, m_k)$ is the current partition of n), or to one of several running sums corresponding to $j = \max [m, k]$ for the case of monotone subsequences of greatest length when this length happens to be j.

These exact distributions were calculated on a 7094 using the multiple-precision fixed point routines described in [2].

The results are shown in Table 1, giving the distribution of permutations of $(1, 2, \cdots, n)$ which contain increasing subsequences of greatest length as well as the distribution according to monotonic subsequences of greatest length. According to the statement of a celebrated theorem of Erdös [8], every sequence of length $k^2 + 1$ contains a maximal monotone subsequence of length at least $k + 1$. Hence the zeros as the first k entries in Table 1b.

The exact calculation for the distribution of monotonicities could be carried out on the 7094 only for sequences of length $\leqq 36$. For sequences of greater length, we examined the expected length of the monotonically increasing subsequences of greatest length by a Monte Carlo procedure. That is, using a pseudo-random number generator we generated sequences of real numbers x $(0 < x < 1)$ with uniform distribution, and verified that the distribution for sequences of length $\leqq 36$ matched the exact distributions. We then examined the distributions in Monte Carlo fashion for sequences of length up to 10,000. The results are summarized in Fig. 1 (and Table 2). It may be observed that the expected length of the spine, over the range $n \leqq 10{,}000$, approaches the value $L_n = 2\sqrt{n}$. This gives us a standard for estimating efficiencies of sorting algorithms which may be applied over P_n.

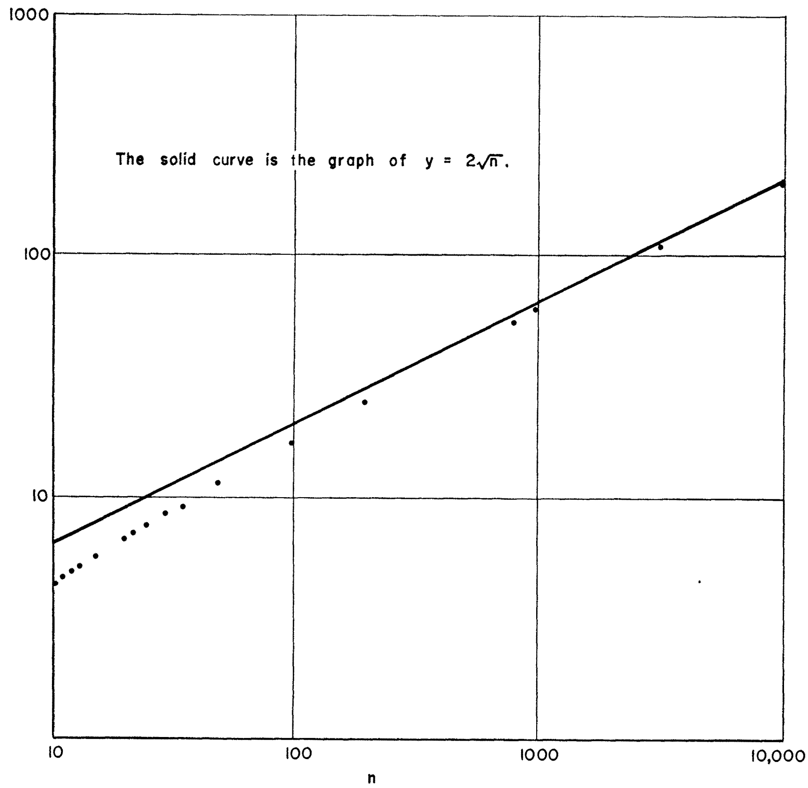


FIGURE 1. Mean values of lengths of monotone increasing subsequences of greatest length over the space of permutations of length n, using Monte Carlo trials on sequences of pseudo-random x, $0 < x < 1$.

TABLE 1a

Distributions of maximal monotone increasing subsequences over the space of permutations of $(1, 2, \cdots, N)$

N	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	1
2	13	41	131	428	1429	4861	16795	58785	208011	742899	2674439
3	9	61	381	2332	14337	89497	569794	3704504	24584693	166335677	1145533650
4	1	16	181	1821	17557	167449	1604098	15555398	153315999	1538907304	15743413076
5		1	25	421	6105	83029	1100902	14516426	192422979	2579725656	35098717902
6			1	36	841	16465	296326	5122877	87116283	1477363967	25191909848
7				1	49	1513	38281	874886	18943343	399080475	8312317976
8					1	64	2521	79861	2250887	59367101	1508071384
9						1	81	3961	153341	5213287	164060352
10							1	100	5941	275705	11110464
11								1	121	8581	469925
12									1	144	12013
13										1	169
14											1

N	15	16	17	18	19
1	1	1	1	1	1
2	9694844	35357669	129644789	477638699	1767263189
3	8017098273	56928364553	409558170361	2981386305018	21935294881644
4	164161815768	1744049683213	18865209953045	207591285198178	2321616416280982
5	485534447114	6835409506841	97966603326993	1429401763567226	21226755241285022
6	434119587475	7583461369373	134533482045389	2426299018270338	44506885647682026
7	172912977525	3615907795025	76340522760097	1631788075873114	35378058306185002
8	37558353900	927716186325	22904111472825	568209449266202	14216730315766814
9	4927007100	143938455225	4142847526101	118504614869214	3389618010035458
10	410474625	14353045401	484748595081	16029615164446	523952747921310
11	22128576	947236425	38094121561	1470147102730	55235843005474
12	766221	41662441	2043822961	93574631242	4087226730670
13	16381	1203441	74797417	4166173834	215285274766
14	196	21841	1830561	128922442	8088065845
15	1	225	28561	2708305	214496074
16		1	256	36721	3910885
17			1	289	46513
18				1	324
19					1

N	20	21	22
1	1	1	1
2	6564120419	24466267019	91482563639
3	162951791097669	1221201051018189	9225637750090023
4	26362085777156567	303635722412859447	3544040394934246209
5	320692032888290224	4926576077469905280	76913478420068425515
6	830512607486659272	15764082963927084216	304295666452406076997
7	778860477345867008	17423197016288134608	396169070839236609236
8	359666061054003144	9216708503647774264	239524408949706575548
9	97376389179852540	2818543211543628620	82388635477750176388
10	17044414451764396	554568496974014588	18113988555378974988
11	2041043061768988	74723654734001996	2725298085020712539
12	172898075436668	7159734192739823	292523675918642499
13	10556881208783	498970549878348	22987539301199111
14	468217205543	25597176520323	1342940739033279
15	15050633363	969739466151	58746150699644
16	345989051	26985380213	1924101911964
17	5527861	543028571	46819431044
18	58141	7666121	831771172
19	361	71821	10451981
20	1	400	87781
21		1	441
22			1

N	23	24	25
1	1	1	1
2	343059613649	1289904147323	4861946401451
3	70209505971502533	537934326588404973	4147337689049888701
4	41881891423602685193	500690223797206725847	6050434838705784406551
5	1219520974164038038455	19625674731276275749737	320348783206253047567401
6	5971518739677370493811	119087070548532813807947	2412379484535302037916891
7	9157097111888617643722	215143361542096212159897	5136996996820257019680841
8	6317740398995612513164	169207499997274346326579	4602911809939402715164066
9	2436180769576352799396	72958306889459609898731	2214789502139053994814716
10	595604303387826752023	19755504320385394380715	662039152774864970449891
11	99428080999387084396	3640046755032713093843	134046618461374364168411
12	11861492339537464775	479491040778079234419	19390512879546189292111
13	1040829060468117119	46606795474062898831	2073904870905091148211
14	68403703794493420	3411981675691129279	167759928098294047111
15	3401595419412851	190527252376282087	10415612075920450911
16	128468428322048	8168500125865719	500718113865946741
17	3675343389664	269170287475509	18709292281579191
18	78885640857	6786436811013	542824153010541
19	1246532772	129464551053	12155099539941
20	14033405	1831706109	207490079241
21	106261	18582345	2643997501
22	484	127513	24297201
23	1	529	151801
24		1	576
25			1

N	*26*	*27*	*28*
1	1	1	1
2	18367353072151	69533550916003	263747951750359
3	3215988926907949542 6	2507174689667939702 02	1964346822230479484023
4	7385238238254585873712 6	9099431776322201992630 42	11310232116090782070465841
5	53002922126526107349285 66	888329318847702519577000 42	150720297902552381432563714 6
6	496165813043994466354382 26	1035620403406365159860967667	2192582635484406531591689725 6
7	1246194422581288791785246 66	3070563111316053793393797817	7681593418929463514096148358 6
8	1271853862638774389877279 16	3569565523674458604017507932	10174407136190020423916308417 6
9	681922234593368134484981 66	2130316698942051016198015702	6753877763023612982350632022 6
10	224415794086234860329509 36	770150147621700978419561077	2677476689247352166400634771 6
11	497468478836604211964986 6	186321569087821491256958077	705072255353796169934109124 6
12	786573753963611623878186	32074009903173294316954102	131691918191241428881727533 6
13	92050364310704637832186	4087354609490748901647222	18199507361203997233037962 6
14	8172068412490464108186	395979688058468617286752	1914653813409462586460578 6
15	559546105197085784116	29697830210588581628302	156380096299085909154574 6
16	29864196368967346036	1745384459871184579552	10051320425021601202006 6
17	1249784830144144516	80996819431842935752	513010163570412335812 6
18	41078307871752516	2978887120344535882	20901857653710268492 6
19	1057716176138766	86826165506931592	6811596755141818486
20	21179941102126	1997955335068642	177313615826634406
21	325456288366	35994624514642	3668913368754571
22	3755016526	500568628717	59792609043091
23	31405401	5254253902	756189714295
24	179401	40166101	7252484527
25	625	210601	50873005
26	1	676	245701
27		1	729
28			1

N	29	30	31
1	1	1	1
2	1002242216651367	3814986502092303	14544636039226908
3	15462158343386281664951	122238896672891001069665	970325456637821942142140
4	141740653186926189324638372	1790036582998939530743648877	22770455209205915603060025597
5	25871893550358473180365541399	449044243619862872721423598179	7876110503741225362359167480199
6	470631631288386313363280613284	10236819433951393776243660748875	225527222131530886314493531876530
7	1950390040214833258890097740134	50241067877038219983230124657600	1312475888601307252438342452786525
8	2944633130025648248522014500254	86511371455863277882723853476200	2579359193231328192634102461101400
9	2173205974659557190038205835604	70971582765623356071324810857700	2352171204427805439293130821881500
10	943409440670609713571889909584	33700117351593715495661064101700	1220678270222172955204768869925305
11	269805363161880496284204974114	10447178628714722178634866396630	409537040100177324029704062492225
12	54517248993798658282931531009	2277900847905046253535807880680	96144246401064295441641257152680
13	8141308676556818373762688529	366440157064983378222220318530	16615605064690814678338772535330
14	926170396512403010651743634	44912755712412555783652789980	2187028591109076615105275286855
15	81973005417740643915053114	4289203871330156652985437480	224524380247140395473550730525
16	5728775881849435165638734	324301002215082697285357800	18286697815711493218117882080
17	319376306355915037772354	19633107355949074371195000	1196174724725300729578276800
18	14297487439649796716054	959064229546178387532600	63388140573056826813361425
19	515761997928294967934	37982369568044622191625	2737060291816929872883300
20	15001117556281650059	1222055891584247185425	96614587298003699934909
21	350943858097699574	31925927141978856309	2790511127801226016125
22	6565817250924743	675007128155925069	65853587607447398904
23	97270960754675	11475430101232224	1264668868266067329
24	1123637013101	155228816648544	19624301369534804
25	9885634319	1644397829384	243364536846276
26	63857305	13319151176	2372835665009
27	285013	79490741	17752922056
28	784	328861	98188781
29	1	841	377581
30		1	900
31			1

N	*32*	*33*	*34*
1	1	1	1
2	55534064877048197	212336130412243109	812944042149730763
3	77319016990773945876877705	6183265315569797796404684 1	496160667633390522403119618
4	29163436719774350072935642 6685	37591802753004459990221961 65965	48750519654702845837670090343690
5	13952939128147295789884692255 4177	24953573185343674567566673532 07729	45030549929498371344718980159549618
6	50301084137779228878072402069 08229	11352687815752540142449327573 0864805	25915900949165093021943507341757 23642
7	34757316644086125833855818210 629081	93272123641741117894541648887 1094393	25353527824408968292697801668489 332618
8	78021383414618228991895594624 384125	23935290945119133462366805084 22151237	74446327825109871020936280154425 059442
9	79103234852498901475016639807 359305	26988667851717132186899262322 32182145	93397918235878046419269724247743 498698
10	44838509270283683161801441967 761005	16702845320694555176762435678 07176205	63096075267224041401539583781897 117650
11	16258866539223347142486988285 081905	65388472949347236735671809745 6295825	26643987678124309733086378500759 601170
12	41018891287879441193688358368 05205	17698592313401084023465852245 4420485	77260716906210995770155538274191 59450
13	75973419344973248743365128332 8585	35057851502535900625566612868 833945	16336742631377362244819109759308 36130
14	10708771526335103136872461754 5005	52785223553845733522830395984 35085	26216358778558077431372210082906 7050
15	11779377563652142010462547688 305	62030509225899379128802488165 2625	32828599452830145581256610566628 170
16	10295558586015723104170909684 05	57988267420258090595629883010 105	32727292690904907862740742680406 50
17	72449327995460834078487733305	43731245318322766169383101018 45	26361284505975260084511212156177 4
18	41439276353797066646473784 25	26880125745337845996522350088 9	17346036203890648128109387808046
19	19394437580629667083302600 9	13567132517272171232545253241	94002460661195691567058605472 2
20	7458944916310495249735449	5651607889444001270864169 69	42199282317113259559913583810
21	2362208036980442534044 89	1948953231720939371313154 5	1575292943311758744546072386
22	61601627905659394252 89	557008975918218063593513	489998813055028279053677 46
23	131990124545762873513	131828325355680927975 77	127041348047397967720869 8
24	23128197235452858 65	2576353741713510283 37	274146073411407125168 58
25	3289164463878462 5	413616337561918436 1	49072110332569725362 6
26	375325561053745	5411075073569844 9	724520271415202111 4
27	3379471497105	570076699965873	87490246706863122
28	23426779857	4754933730273	853666255814802
29	120413921	30626645457	6614681604114
30	431521	146679105	39691351954
31	961	491041	177551265
32	1	1024	556513
33		1	1089
34			1

N	*35*	*36*
1	1	1
2	3116285494907301261	11959798385860453491
3	3994086211355939463493383548	32249846192152237908327459893
4	635848553797809170295520418710782	8338432923207513905107942840897207
5	819588244871579975094742632650930358	15038912583528984220392081381933578208
6	59812066183117095755447194860322648806	1395020405364298522836760442796104706656
7	697812273978076490697755421027696727478	19439642570069266048240891724790133009888
8	2346831372517025163776957078701792456870	74956422241816331218926228837298911186720
9	3277611718424868517776769808161439844462	116608638100360116710442324796876756408352
10	2416847811147063808748488120922687920734	93859283446101269401691563039073753552064
11	1100086329884899081144637854613456703390	46025756664927526758904180662495739416960
12	341328852295630743648369780452363294190	15264266027718982566224773594719844868640
13	76917535286353434994167211570562088390	3660517378342559966873860780713566493240
14	13129491403244799666927129567113776890	663449927112151517704628177014442337840
15	1747879793744666872732913487156814290	93703493478435896513152031028151961064
16	185330636189134919022484650048294414	10542497140124782901363704728467537040
17	15896803965499693938651780896898834	960400377652127813719842557392971204
18	1116003041710932482624313323727894	71716805772727233527588176423578804
19	64686278939242244792776254311634	4430941771361782282147424398422404
20	3115625942782130518017444226214	228095611162787120114225791488548
21	125263010041612539140025735398	9833255433124590718722650114864
22	4215653028099462112368819038	356237120025927944601115628528
23	118905644693483891716868358	10866695854770117187576414928
24	2809856400891651324334238	279258361123714921066189328
25	55518761604460803409598	6040431285173037746028992
26	913685147570122124342	109700112356207611240448
27	12448894244607115902	1665622494029241660512
28	139195797135863382	21008924568821050912
29	1261485212494102	218144396451607687
30	9104618690917	1841133016223551
31	51020199826	12407725834267
32	213654981	65081293411
33	628321	255676261
34	1156	706861
35	1	1225
36		1

TABLE 1b

Distributions of maximal monotone subsequences over the space of permutations of $(1, 2, \cdots, N)$

N	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
1	0	0	0	0	0	0	0	0	0	0	0
2	4	0	0	0	0	0	0	0	0	0	0
3	18	86	306	882	1764	1764	0	0	0	0	0
4	2	32	362	3242	24564	163872	985032	5323032	25038288	97173648	288576288
5		2	50	842	12210	161158	1969348	22493900	241860366	2461364334	23779896712
6			2	72	1682	32930	592652	10182250	169395678	2741446006	43213948960
7				2	98	3026	76562	1749772	37886686	797307174	16528420624
8					2	128	5042	159722	4501774	118734202	3016142768
9						2	162	7922	306682	10426574	328120704
10							2	200	11882	551410	22220928
11								2	242	17162	939850
12									2	288	24026
13										2	338
14											2

N	*15*	*16*	*17*	*18*	*19*
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	577152576	577152576	0	0	0
5	217908143024	1882013490500	15137332742532	111495773096136	736543205438040
6	663703605292	9933107320916	144868857114644	2058529251860328	28477732544688960
7	339659751732	6934946649572	140700564230692	2834419897105696	56633617912329112
8	75104929176	1853582024436	45645218200032	1125758970041192	27855001516505912
9	9854014200	287876910450	8285529415302	236974544144628	6775208774870652
10	820949250	28706090802	969497190162	32059230328892	1047903131938220
11	44257152	1894472850	76188243122	2940294205460	110471686010948
12	1532442	83324882	4087645922	187149262484	8174453461340
13	32762	2406882	149594834	8332347668	430570549532
14	392	43682	3661122	257844884	16176131690
15	2	450	57122	5416610	428992148
16		2	512	73442	7821770
17			2	578	93026
18				2	648
19					2

N	*20*	*21*	*22*
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	4254132236622840	20794481008587000	82244764498782600
6	38290251723 1190424	498901184168750 9064	627115314902906463 92
7	112108858638020 2928	2196365406131114 8288	4254425323457184 72872
8	69170955086648 9296	1723066810741081 0776	4301649242480701 69032
9	19440906673340 8704	5612953233636388 664	1632972991764710 00376
10	34088191453068 288	1109042572932268 208	3621778136781251 9128
11	4082086123537976	149447275333224456	5450584636999631766
12	345796150873336	14319468385479646	585047351837284998
13	21113762417566	997941099756696	45975078602398222
14	936434411086	51194353040646	2685881478066558
15	30101266726	1939478932302	117492301399288
16	691978102	53970760426	3848203823928
17	11055722	1086057142	93638862088
18	116282	15332242	1663542344
19	722	143642	20903962
20	2	800	175562
21		2	882
22			2

N	*23*	*24*	*25*
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	245804974123480200	491609948246960400	491609948246960400
6	7559710897271781270888	8676145262210193012960	94003717658533097844000
7	81400329329241640320880	153675090362576773465488	285913331823104906399880
8	107505115684549975691920	2686377368800891199209280	6704129404938223686722000
9	47901432782154891013360	14167749832361808509615880	4223257115141515056146350
10	11903095089297245796540	3944221148745271925862200	13193371977206438081943500
11	198854038310659029760	727981043384889002174200	2680626176773117639067500
12	23722984181440622926	95898187551482076372600	387809795737239914017500
13	2081658120936234238	9321359094812579766200	41478097344977226240860
14	136807407588986840	682396335138225855800	3355198561965880942220
15	6803190838825702	38105450475256417400	208312241518409018220
16	256936856644096	1633700025173143800	10014362277318934820
17	7350686779328	53834057495101800	374185845631583820
18	157771281714	1357287362202600	10856483060210820
19	2493065544	25892910210600	243101990798820
20	28066810	366341221800	4149801584820
21	212522	3716469000	52879950020
22	968	25502600	485944020
23	2	105800	3036020
24		2	1152
25			2

N	26	27	28
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	95238202785319485331440 0	892559210111330129982900 0	764255730554665530085476 00
7	5234194002448571370245880 0	941037757041119409946427400	16575282453100233870595785600
8	1669073387058009817518020 00	414109520954784824710451400 0	10229018872507956516144245400 0
9	12678703040113536326223760 0	382996532320524137230246260 0	11630081221168227619375032930 0
10	4458149146064005218551420 0	152226457350336276483383100 0	5252319574784846704103876550 0
11	994651218130515540889690 0	37240422234580371091196970 0	1408308750104610791825360010 0
12	157314005909391488215050 0	6414705029734399338627510 0	263373004342730653819569630 0
13	18410072497841975109243 6	817470844203532966156897 2	36398995970991764118907467 6
14	1634413682498092821637 2	79195937600876475421350 4	3829307620433327049139157 2
15	111909221039417156823 2	5939566042117716325660 4	312760192598171818309149 2
16	5972839273793469207 2	349076891974236915910 4	20102640850043202404013 2
17	249956966028828903 2	16199363886368587150 4	1026020327140824671625 2
18	8215661574350503 2	595777424068907176 4	41803715307420536985 2
19	211543235227753 2	17365233101386318 4	1362319351028363697 2
20	4235988220425 2	399591067013728 4	35462723165326881 2
21	65091257673 2	7198924902928 4	733782673750914 2
22	751003305 2	100113725743 4	11958521808618 2
23	6281080 2	1050850780 4	151237942859 0
24	35880 2	8033220 2	1450496905 4
25	125 0	42120 2	10174601 0
26	2	135 2	49140 2
27		2	145 8
28			2

N	*29*	*30*	*31*
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	589207733618083635076678800	4017834614129468263414826400	23692354719303941547720386400
7	285202153278958513785707520000	4777672028403661379299865820600	77623700024019150137247624840600
8	2513137284768128335678924779600	61354192232985708594062524631400	1486899898340427286368998222579400
9	3546523725021576401940276722100	108498874067996587999017464032200	3326862986586311241555650840434200
10	1830683985233926033276259240700	64426925473775882732381174069400	2287930539402346328092953064629900
11	538290932005594148436878681700	20804390394983947746227177299800	813200022570515550808858717299300
12	109023704504845696062407712600	4554817493732644750790195030400	192204847001779358208731530304400
13	16282588447793719958338703304	732876506258667437085324970680	33230764580423680716981408825480
14	1852340772834412694565585668	89825506876602139897193098560	4374056362868704012595427126060
15	163946010833871946234546228	8578407741549180952620369360	449048760085161503360103812100
16	11457551763698870331277468	648602004430165394570715600	36573395631398924991224813760
17	638752612711830075544708	39266214711898148742390000	2392349449450601459156553600
18	28594974879299593432108	1918128459092356775065200	126776281146113653626722850
19	1031523995856589935868	75964739136089244383250	5474120583633859745766600
20	30002235112563300118	2444111783168494370850	193229174596007399869818
21	701887716195399148	36851854283957712618	5581022255602452032250
22	13131634501849486	1350014256311850138	131707175214894797808
23	194541921509350	22950860202464448	2529337736532134658
24	2247274026202	310457633297088	39248602739069608
25	19771268638	3288795658768	486729073692552
26	127714610	26638302352	4745671330018
27	570026	158981482	35505844112
28	1568	657722	196377562
29	2	1682	755162
30		2	1800
31			2

N	*32*	*33*	*34*
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	1172398087983769410901717008 00	466537502959586314018384564800	1397195216117310308303763100800
7	12180287885496809170034392081632 00	1837456725622881003170025120281 2800	2651701765919760875383672242402432 00
8	35731933969901471743292605066276800	8504414821016755102403194067606016 00	20019612952745687325073221493720761600
9	102149478451453503725655595992317700	313799832659038146612253279150713 2100	96366404707773821179385595553103329800
10	81927367586821122917342698830532500	2955913746239917536402680534032391700	1073708839212091740501954118168862970 00
11	32147481661001074987395360190510500	1285098171830687847235480636082667300	519325761364934619697068539947049010 00
12	8197061613892990320149108104590900	353456937224263313066214631017011700	1541414891711286888099328686858195780 0
13	1519420860841425827079577453336260	701109945171577702788072224746012 20	3266909218026167211860801160561355080
14	214175304882927153555062226802260	105570276606840415742248875358960 20	524325073480942763900955522492791400
15	23558755020416536108098219522660	124061016226972018922687894020170 0	65657194977601776429750506532740640
16	2059111717183930993052820598260	11597653483236550492157228823216 0	6545458535736369512361315896644200
17	144898655990921668156975466610	874624906366419193624137765159 0	527225690119174755622602852617748
18	8287855270759413329294756850	537602514906756919930447001778	34692072407781296256218775616092
19	387888751612593341666052018	27134265034544342465090506482	1880049213223913831341172109444
20	14917889832620990499470898	1130321577888800254172833938	83498564634226519119827167620
21	472441607396088506808978	38979064634418787426263090	3150585886623517489092144772
22	12320325581131878850578	1114017951836436127187026	97999762611005655810735492
23	263980249091525747026	26365665071136185595154	2540826960947959354417396
24	4625639447090571730	515270748342702056674	54829214682281425033716
25	65783289277569250	8272326751238368722	981442206651394507252
26	750651122107490	108221501471396898	14490405428304042228
27	6758942994210	1140153399931746	174980493413726244
28	46853559714	9509867460546	1707332511629604
29	240827842	61253290914	13229363208228
30	863024	293358210	79382703908
31	1922	982082	355102530
32	2	2048	1113026
33		2	2178
34			2

N	*35*	*36*
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	27943904322346206166075262016000	279439043223462061660752620160 0
7	364130337146417014038128944409401920 0	4730238587675126294079168036478965120 0
8	465398762365262092373752560180569107200	10666155857406751349088621857909306035200
9	2956033630424242850168930272742455579800	9050367408659923553800860794428837698480 0
10	3923448021404298815368599763876489462200	14411070427168582707423940583168340010800 0
11	2120729276021540978123490370436277859800	8747091527099545887212541834170507840400 0
12	679941588966377982290083835615230414200	3033942620568527115531516150338050915920 0
13	153796056449988057049755243386096817240	731770910349313429684473913808748407484 0
14	26258742863974898853251857892785234080	132687413768104388999821143934835305548 0
15	3495758975783663056219222404208369080	18740690067748451831329589775820383892 8
16	370661271796094690099583907209585528	2108499416301961203016251845767391448 0
17	31793607930839377511589787895627868	192080075524965054161539796756279860 8
18	2232006083421859519530636624767388	14343361154544881066857741983660820 8
19	129372557878484489585552508623268	886188354272356456429484879684480 8
20	6231251885564261036034888452428	45619122232557424022845158297709 6
21	250526020083225078280051470796	1966651086624918143744530022972 8
22	8431306056198924224737638076	71247424005185588920223125705 6
23	237811289386967783433736716	2173339170954023437515282985 6
24	5619712801783302648668476	55851672224742984213237865 6
25	111037523208921606819196	1208086257034607549205798 4
26	1827370295140244248684	21940022471241522248089 6
27	24897788489214231804	33312449880584833210 24
28	278391594271726764	4201784913764210182 4
29	2522970424988204	43628879290321537 4
30	18209237381834	36822660324471 02
31	102040399652	24815451668534
32	427309962	130162586822
33	1256642	511352522
34	2312	1413722
35	2	2450
36		2

3. Two Sorting Strategies over P_n. We consider first the 0th order sorting strategy over P_n. This consists of selecting the initial element (in a permutation sequence), and, having selected an element, selecting the first following element in the sequence which is greater. The resulting selected subsequence is called the *distinguished subsequence.*

3.1. THEOREM. *In the space P_n the expected length of the distinguished subsequence is*

$$L_n = \sum_{k=1}^{n} k^{-1}.$$

Proof. The statement of the theorem is clearly true for $n = 1$. We proceed by induction. Suppose that the statement of the theorem holds for $k \leqq n$. Given any particular sequence in P_n, there are $n + 1$ ways in which sequences of P_{n+1} may be formed through the adjunction of the number $n + 1$ to this particular sequence, and with each of these new sequences is associated the probability $(n + 1)^{-1}$. For each new sequence so constructed, the corresponding distinguished subsequence ends with the adjoined number. Hence the expected length of the corresponding distinguished subsequence is greater by unity than the expected length of a distinguished subsequence in that part of the original sequence preceding the adjoined integer, averaged over the positions that the adjoined integer may occupy. This is just

$$L_{n+1} = (n + 1)^{-1} \sum_{k=0}^{n} L_k + 1$$

where L_0 is taken to be zero. It follows that

$$L_{n+1} = L_n + (n + 1)^{-1}.$$

This concludes the proof.

We now consider the nth order strategy for natural sorting in P_n. According to the definition, the sorting algorithm of an nth order strategy has available, at the point where it decides whether to select the ith sequence element, the identity of the previously considered sequence elements x_k, $k < i$. The selection algorithm for P_n accordingly has the following form. A choice level C_1 (where C_1 is a natural number, $0 < C_1 \leqq n$) determines the selection of the first sequence element x_1 (i.e. if $x_1 \leqq C_1$ then x_1 is selected). A selection level $C_2 = f_1(x_1)$ is defined, depending upon x_1. The second sequence element x_2 is selected accordingly (if $x_1 \leqq x_2 \leqq C_2$ in the case that x_1 was selected, or else simply $x_2 \leqq C_2$ in the alternate case). What is required is the sequence of functions $f_0(S) = C_1$, $f_1(S) = C_2$, $\cdots$ (where S is an arbitrary sequence in P_n) which maximizes the expected length of the selected subsequence.

To show how the choice levels are made, we will consider the example of P_4. We suppose that the expected lengths from the optimal choice functions are known for P_1, P_2, and P_3. (These expected lengths are: $\langle L_1 \rangle = 1$, $\langle L_2 \rangle = 3/2$, $\langle L_3 \rangle = 2$.) There are four possibilities for the value of C_1 (i.e. $C_1 = 1, 2, 3$, or 4). Consider the expected lengths of the selected subsequence which results from each possible value of C_1. Thus, if $C_1 = 1$ the first element of a permutation sequence will be selected if and only if it has the value unity. In this case the remaining sequence consists of a permutation of the sequence (2, 3, 4) and by, assumption, the strategy and expected length of P_3 are known and may be applied to the permutation of (2, 3, 4)—with certain obvious adjustments. The expected length in this case, which we shall denote $L_4(1)$, is just the value of $L_3 + 1$ weighted by the probability (which is 1/4) that unity occurs as the initial entry of the sequence from P_4 added to the value $\langle L_3 \rangle$ which in turn is weighted by the probability that unity does not occur as the first sequence entry. So, $L_4(1) = 4^{-1}[(1 + \langle L_3 \rangle) + 3\langle L_3 \rangle]$.

TABLE 2

Expected lengths of (1) monotonically increasing subsequences of greatest length, (2) of monotone subsequences of greatest length, and (3) observed (Monte Carlo) means of monotonically increasing subsequences of greatest length, (4) of monotone increasing subsequences selected according to the nth order strategy; (5) is the initial selection level used in (4).

N	(1)	(2)	(3)	(4)	(5)
3	2.000	2.333			
4	2.416	2.916			
5	2.791	3.300		2.73	3
6	3.140	3.650	3.155	3.04	3
7	3.465	4.021	3.446	3.33	3
8	3.770	4.350	3.760	3.60	4
9	4.059	4.647	4.049	3.86	4
10	4.334	4.938	4.333	4.10	4
11	4.598	5.222	4.577	4.32	4
12	4.852	5.490	4.849	4.54	5
13	5.096	5.745	5.115	4.75	5
14	5.332	5.991	5.323	4.95	5
15	5.561	6.232	5.560	5.15	5
16	5.783	6.465	5.786	5.33	5
17	5.999	6.691	6.013	5.51	5
18	6.209	6.910	6.213	5.69	6
19	6.414	7.123	6.417	5.86	6
20	6.614	7.332	6.618	6.03	6
21	6.810	7.536	6.832	6.19	6
22	7.002	7.736	7.002	6.35	6
23	7.189	7.931	7.165	6.50	6
24	7.373	8.122	7.354	6.65	7
25	7.554	8.309	7.556	6.80	7
26	7.731	8.493	7.741	6.94	7
27	7.905	8.673	7.926	7.09	7
28	8.076	8.851	8.074	7.23	7
29	8.244	9.025	8.258	7.36	7
30	8.410	9.196	8.420	7.50	7
31	8.573	9.365	8.553	7.63	8
32	8.734	9.531	8.717	7.76	8
33	8.892	9.695	8.871	7.87	8
34	9.049	9.857	9.026	8.01	8
35	9.203	10.016	9.206	8.14	8
36	9.355	10.173	9.368	8.26	8
100			16.723	14.05	14
200			24.508	20.02	20
1000			58.154	44.99	44
10000			192.2	142.07	141

If we take $C_1 = 2$ then the argument proceeds in a similar way, except that now if either 1 or 2 occurs as the initial sequence element it is selected, and if 2 occurs and is selected then since the selection algorithm applies now only to sequence entries with values greater than 2, the strategy and expected length of the selected subsequence from P_2 comes into play, thus

$$L_4(2) = 4^{-1}[(1 + \langle L_3\rangle) + (1 + \langle L_2\rangle) + 2\langle L_3\rangle] .$$

And similarly $L_4(3)$ and $L_4(4)$ may be evaluated. Then C_1 is defined as that value of k which maximizes $L_4(k)$. In the general case

$$\langle L_{n+1}\rangle = \max_{0<k\leqq n} (n + 1)^{-1}\left[k + (n + 1 - k)\langle L_n\rangle + \sum_{i=2}^{n} \langle L_{n-i+1}\rangle\right]$$

and it follows that the expected lengths of the selected subsequences and the choice levels are determined at the same time.

The required functions may be computed, using a course-of-values recursion. For this we are indebted to Mr. David Matula, who provided us with columns 4 and 5 of Table 2, computed in double-precision directly from the above equation. It will be observed that the expected length of the selected monotone subsequences, in sequences of length n, is approximately $(2n)^{1/2}$.

Conclusions. Over the range which has been examined ($n \leqq 10{,}000$), let r_1 be the ratio of the expected value of a monotone subsequence, selected according to the 0th order strategy, to the expected length of the monotone subsequence of greatest length. Then

$$r_1 \sim \frac{\sum_1^n k^{-1}}{2\sqrt{n}} \sim \frac{\log n}{2\sqrt{n}} \to 0 .$$

Let r_2 be the ratio of the expected length of the monotone subsequence, selected according to the nth order strategy, to the expected length of the monotone subsequence of greatest length. Then

$$r_2 \sim (2n)^{1/2}/2\sqrt{n} = 1/\sqrt{2} .$$

TABLE 3
Tableaux with maximal weights for group of order n; $10 \leqq n \leqq 36$

order 10	*order 11*	*order 12*	*order 13*	*order 14*	*order 15*
7 5 3 1	8 6 4 2 1	9 6 4 2 1	9 6 4 3 1	10 7 5 4 2 1	9 7 5 3 1
5 3 1	5 3 1	6 3 1	7 4 2 1	7 4 2 1	7 5 3 1
3 1	3 1	4 1	4 1	4 1	5 3 1
1	1	2	2	2	3 1
		1	1	1	1

order 16	*order 17*	*order 18*	*order 19*	*order 20*
10 8 6 4 2 1	11 8 6 4 2 1	12 9 7 5 3 2 1	12 9 7 5 4 2 1	12 10 7 5 4 2 1
7 5 3 1	8 5 3 1	8 5 3 1	9 6 4 2 1	9 7 4 2 1
5 3 1	6 3 1	6 3 1	6 3 1	6 4 1
3 1	4 1	4 1	4 1	4 2
1	2	2	2	3 1
	1	1	1	1

TABLE 3—*Continued*

order 21
```
13 10 7 5 4 2 1
10  7 4 2 1
 7  4 1
 5  2
 4  1
 2
 1
```

order 22
```
12 10 8 6 4 2 1
 9  7 5 3 1
 7  5 3 1
 5  3 1
 3  1
 1
```

order 23
```
13 10 8 6 4 2 1
10  7 5 3 1
 8  5 3 1
 6  3 1
 4  1
 2
 1
```

order 24
```
14 11 9 7 5 3 2 1
10  7 5 3 1
 8  5 3 1
 6  3 1
 4  1
 2
 1
```

order 25
```
14 11 9 7 5 4 2 1
11  8 6 4 2 1
 8  5 3 1
 6  3 1
 4  1
 2
 1
```

order 26
```
15 11 9 7 5 4 2 1
12  8 6 4 2 1
 9  5 3 1
 7  3 1
 5  1
 3
 2
 1
```

order 27
```
15 12 9 7 5 4 2 1
12  9 6 4 2 1
 9  6 3 1
 7  4 1
 5  2
 4  1
 2
 1
```

order 28
```
15 12 9 7 6 4 2 1
12  9 6 4 3 1
10  7 4 2 1
 7  4 1
 5  2
 4  1
 2
 1
```

order 29
```
14 12 10 8 6 4 2 1
11  9  7 5 3 1
 9  7  5 3 1
 7  5  3 1
 5  3  1
 3  1
 1
```

order 30
```
15 12 10 8 6 4 2 1
12  9  7 5 3 1
10  7  5 3 1
 8  5  3 1
 6  3  1
 4  1
 2
 1
```

order 31
```
16 13 11 9 7 5 3 2 1
12  9  7 5 3 1
10  7  5 3 1
 8  5  3 1
 6  3  1
 4  1
 2
 1
```

order 32
```
13 16 11 9 7 5 4 2 1
13 10  8 6 4 2 1
10  7  5 3 1
 8  5  3 1
 6  3  1
 4  1
 2
 1
```

order 33
```
17 13 11 9 7 5 4 2 1
14 10  8 6 4 2 1
11  7  5 3 1
 9  5  3 1
 7  3  1
 5  1
 3
 2
 1
```

order 34
```
17 14 11 9 7 5 4 2 1
14 11  8 6 4 2 1
11  8  5 3 1
 9  6  3 1
 7  4  1
 5  2
 4  1
 2
 1
```

order 35
```
17 14 11 9 7 6 4 2 1
14 11  8 6 4 3 1
12  9  6 4 2 1
 9  6  3 1
 7  4  1
 5  2
 4  1
 2
 1
```

order 36
```
17 14 12 9 7 6 4 2 1
14 11  9 6 4 3 1
12  9  7 4 2 1
 9  6  4 1
 7  4  2
 6  3  1
 4  1
 2
 1
```

Open questions. This study has raised some questions of which the following, in the area of asymptotic combinatorics, are particularly interesting:

(1) What is the asymptotic form of the distribution of spines in the case of permutation spaces?

Presumably an analytical derivation of the answer to (1) would require an answer to

(2) For given n, which Young tableau maximizes the value of the Frame-Robinson-Thrall function? If the maximizing tableau is denoted T_n then

(3) Is there an algorithm which permits the immediate construction of T_{n+1} from T_n?

In computing the exact distributions of the spines (for $n \leqq 36$), the corresponding T_n were obtained automatically. These are exhibited (for $10 \leqq n \leqq 36$) in Table 3, although it is not believed that they suggest the form of T_n for large n.

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1. R. M. Baer & P. Brock, "Natural sorting," *J. Soc. Indust. Appl. Math.*, v. 10, 1962, pp. 284–304. MR **27** #4396.
2. R. M. Baer & M. G. Redlich, "Multiple-precision arithmetic and the exact calculation of the 3-*j*, 6-*j* and 9-*j* symbols," *Comm. ACM*, v. 7, 1964, pp. 657–659. MR **31** #865.
3. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R. I., 1940; rev. ed., 1967.
4. L. Carlitz, "A binomial identity arising from a sorting problem," *SIAM Rev.*, v. 6, 1964, pp. 20–30. MR **29** #4708.
5. L. Carlitz, "Some multiple sums and binomial identities," *J. Soc. Indust. Appl. Math.*, v. 13, 1965, pp. 469–486. MR **31** #2160.
6. P. Erdös, "On some asymptotic formulas in the theory of partitions," *Bull. Amer. Math. Soc.*, v. 52, 1946, pp. 185–188. MR **7**, 273.
7. P. Erdös & J. Lehner, "The distribution of the number of summands in the partitions of a positive integer," *Duke Math J.*, v. 8, 1941, pp. 335–345. MR **3**, 69.
8. P. Erdös & G. Szekeres, "A combinatorial problem in geometry," *Compositio Math.*, v. 2, 1935, pp. 363–470.
9. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, 2nd ed., Wiley, New York, 1957. MR **19**, 466.
10. J. S. Frame, G. de B. Robinson & R. M. Thrall, "The hook graphs of the symmetric group," *Canad. J. Math.*, v. 6, 1954, pp. 316–324. MR **15**, 931.
11. Hansraj Gupta, C. E. Gwyther & J. C. P. Miller, *Tables of Partitions*, Royal Society Mathematical Tables, Vol. IV, Cambridge Univ. Press, New York, 1962.
12. C. Schensted, "Longest increasing and decreasing sub-sequences," *Canad. J. Math.*, v. 13, 1961, pp. 179–191. MR **22** #12047.
13. M. P. Schutzenberger, "Quelques remarques sur un construction de Schensted," *Math. Scand.*, v. 12, 1963, pp. 117–128. MR **32** #7433.
14. G. Szekeres, "An asymptotic formula in the theory of partitions," *Quart. J. Math. Oxford Ser.* (2), v. 2, 1951, pp. 85–108. MR **13**, 210.